

# Action functionals for strings in four dimensions

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## Abstract

*All possible action functionals on the space of surfaces in  $\mathbf{R}^4$  that depend only on first and second derivatives of the functions, entering the equation of the surface, and satisfy the condition of invariance with respect to rigid motions are described.*

It was conjectured that quantum chromodynamics is equivalent to some kind of string theory. The bosonic part of the action functional of such a string theory should be considered as a functional defined on the space of 2-dimensional surfaces in 4-dimensional space. The simplest possible action (Nambu-Goto action = area of the surface in the induced metric) can not describe the QCD string; A. Polyakov [1] suggested to include in the string action a term expressed in terms of extrinsic curvature of the surface in the induced metric.

In the present paper we describe all possible action functionals on the space of surfaces in  $\mathbf{R}^4$  that depend only on first and second derivatives of the functions, entering the equation of the surface, and satisfy the natural condition of invariance with respect to rigid motions. Similar results can be obtained for action functionals defined on the space of surfaces in Minkowski space.

It is assumed that the action functional can be represented in the form:

$$S = \int_{\Gamma} A \left( X(U), \frac{\partial X^\alpha}{\partial U^j}, \frac{\partial^2 X^\alpha}{\partial U^j \partial U^k} \right) dU. \quad (1)$$

Here we denote by  $X^\alpha$  the coordinates in  $\mathbf{R}^4$  and by  $U^j$  the parameters of the 2-dimensional surface  $\Gamma$ . Of course, the integral (1) should be independent on the choice of parameterization  $X = X(U)$  of  $\Gamma$ . In this case the function  $A$  is called a *density*. We supposed that the function  $A$  depends on the first and second derivatives of  $X(U)$  only; in this case one says that  $A$  is a density of *rank 2*. We assume the invariance of  $S$  with respect to rigid motions of  $\mathbf{R}^4$ . In particular,  $S$  should be invariant with respect to shifts (parallel transports), therefore  $A$  will not depend on  $X(U)$  itself but only on its derivatives.

There are two possible ways to define the measure of integration  $dU$  in the integral (1): as a positive measure, or as a signed one (it changes sign under a orientation reversing reparametrization). In the second case we will deal with oriented surfaces only. The definition of density above requires  $A$  to be "covariant" under the action of the reparametrization group  $L$ . And for the different definitions of  $dU$  the meaning of the word "covariant" is not the same: in the case of a reparametrization  $U = K(U')$  ( $K \in L$ ) the new density  $A'$  can be obtained from the old one  $A$  by the multiplication either by the Jacobian of the transformation  $K$  (density of the *second kind*), or by the absolute value of this Jacobian (density of the *first kind*). Since the results in these two cases are very similar, we will consider only densities of the first kind.

We will prove that all densities satisfying the requirements above can be expressed in terms of the following four basic invariants:

$$Q_1 = \frac{1}{\Omega} \epsilon^{km} \epsilon^{ln} \delta_{\alpha\beta} D_k \partial_l X^\alpha D_m \partial_n X^\beta, \quad (2)$$

$$Q_2 = \frac{1}{\Omega^2} \epsilon^{ik} \epsilon^{jn} \epsilon^{pl} \epsilon^{qm} \delta_{\alpha\beta} \delta_{\gamma\tau} \delta_{\mu\nu} \partial_i X^\gamma \partial_j X^\tau \partial_p X^\mu \partial_q X^\nu D_k \partial_l X^\alpha D_m \partial_n X^\beta, \quad (3)$$

$$Q_3 = \frac{1}{\Omega^2} \epsilon^{pq} \epsilon^{ik} \epsilon^{jl} \epsilon^{mn} \epsilon_{\alpha\beta\gamma\delta} \delta_{\mu\nu} \partial_i X^\mu \partial_j X^\nu \partial_m X^\alpha \partial_n X^\beta D_p \partial_l X^\gamma D_k \partial_q X^\delta. \quad (4)$$

$$Q_4 = \frac{1}{\Omega^2} \epsilon^{im} \epsilon^{kp} \epsilon^{lq} \epsilon^{jn} \delta_{\alpha\beta} \delta_{\mu\nu} D_i \partial_j X^\alpha D_k \partial_l X^\beta D_p \partial_q X^\mu D_m \partial_n X^\nu. \quad (5)$$

Here  $D_i \partial_j X^\alpha$  are covariant derivatives of the corresponding 1-forms (in parameter space)  $\partial_j X^\alpha$  (for fixed  $\alpha$ ) with respect to the metric  $s_{ij} = \delta_{\alpha\beta} \partial_i X^\alpha \partial_j X^\beta$  induced by the embedding of the surface  $\Gamma$  in  $\mathbf{R}^4$ , and

$$\Omega = \frac{1}{2} \epsilon^{ik} \epsilon^{jl} \delta_{\alpha\beta} \delta_{\mu\nu} \partial_i X^\alpha \partial_j X^\beta \partial_k X^\mu \partial_l X^\nu \quad (6)$$

is the determinant of  $s$ .

To construct a density one should take a real function in four variables  $f(p, q, r, t)$  that is invariant under the transformation  $r \rightarrow -r$  and is regular in the domain specified by the conditions:  $p \geq 0$ ,  $q \geq 0$ ,  $t \geq 0$ ,  $p^2 > 4r^2$ ,  $pqt \geq q^2 r^2 + t^2$ ; then the corresponding density will have the form:

$$A = F(Q_1, Q_2, Q_3, Q_4) \sqrt{\Omega}, \quad (7)$$

where

$$F(Q_1, Q_2, Q_3, Q_4) = f\left(\frac{Q_2 - Q_1}{4}, \frac{Q_2 + Q_1}{4}, \frac{Q_3}{8}, \frac{Q_1^2 + Q_2^2}{32} - \frac{Q_3^2}{128} - \frac{Q_4}{8}\right). \quad (8)$$

We will prove that every reparametrization invariant action functional of the form (1) can be obtained by means of the above construction.

If  $F$  is identically equal to 1, we have the usual *Nambu-Goto* density, having the meaning of the surface area;  $F = Q_1$  corresponds to the *Einstein action functional* (here density is equal to the scalar intrinsic curvature of the surface); if  $F = Q_2$  we obtain the extrinsic curvature density [1].

The proof of these results is based on the ideas of [2, 3]. It is pointed out there, that the density  $A$  can be considered as a function on the space  $\mathcal{Q}$  of all quadratic surfaces, i.e. surfaces of the form:

$$x = B_i u^i + G_{ij} u^i u^j. \quad (9)$$

Really, the density  $A$  in (1) depends only on the first two derivatives of  $X$  and for each point  $X_0$  of  $\Gamma$  (corresponded to the value  $U_0$  of the parameter) we can consider these derivatives as coefficients of a quadratic surface  $\Sigma_{X_0}$ , approximating  $\Gamma$  at this point:

$$B_i = \left. \frac{\partial X}{\partial U^i} \right|_{X_0} = \left. \frac{\partial x}{\partial u^i} \right|_0, \quad G_{ij} = \left. \frac{1}{2} \frac{\partial^2 X}{\partial U^i \partial U^j} \right|_{X_0} = \left. \frac{1}{2} \frac{\partial^2 x}{\partial u^i \partial u^j} \right|_0.$$

Here  $u = U - U_0$ ,  $x = X - X_0$ . We see that  $G$  is a symmetric  $2 \times 2$  matrix with values in  $\mathbf{R}^4$  and  $B$  is a  $1 \times 2$  matrix also with values in  $\mathbf{R}^4$ . Sometimes we will use the more explicit form of this formula:

$$x = au + bv + fu^2 + 2guv + hv^2, \quad (9a)$$

where  $u = u^1, v = u^2$ ,  $a = B_1, b = B_2, f = G_{11}, g = G_{12} = G_{21}, h = G_{22}$  and all the coefficients are vectors in  $\mathbf{R}^4$ .

To describe densities as functions on  $\mathcal{Q}$  we consider the group  $\Lambda$  acting on  $\mathcal{Q}$  and generated by linear and quadratic reparametrizations.

The *linear reparametrization*

$$u^i \rightarrow K_j^i u^j \quad (10)$$

transforms the surface (2) into the surface

$$x = B'_i u^i + G'_{ij} u^i u^j,$$

where

$$B'_i = K_i^j B_j, \quad G'_{ij} = K_i^l K_j^m G_{lm}.$$

The *quadratic reparametrization*

$$u^i \rightarrow u^i + T_{jk}^i u^j u^k \quad (11)$$

transforms (2) into the quadratic surface

$$x = B_i u^i + G'_{ij} u^i u^j,$$

where

$$G'_{ij} = G_{ij} + B_k T_{ij}^k.$$

It is easy to show that the density of rank 2 can be considered as a function  $A$  defined on the space  $\mathcal{Q}$  that is invariant under quadratic reparametrizations and its value after linear reparametrization (10) can be obtained from the initial one by means of multiplication on  $|\det K|$ .

It is not difficult to show, that any surface  $\Sigma \in \mathcal{Q}$  can be reduced to some *standard form*  $\Sigma^0$ :

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ \alpha + \beta \\ \gamma \end{pmatrix} u^2 + 2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu \end{pmatrix} uv + \begin{pmatrix} 0 \\ 0 \\ \beta - \alpha \\ \gamma \end{pmatrix} v^2$$

by means of transformations from the group  $G = SO(4) \times \Lambda$ . Here  $\alpha, \beta, \gamma$ , and  $\mu$  are some arbitrary real numbers. This form is *almost* unique: two standard forms that differs only by the sign of  $\alpha$  correspond to the same surface  $\Sigma$ . So, we will consider  $\alpha^2$  as a parameter of  $\Sigma^{(0)}$  instead of  $\alpha$  itself.

We should notice, that the properties of density described above allow us to reconstruct the value of a density on a surface  $\Sigma \in \mathcal{Q}$  from its value on the corresponding surface  $\Sigma^0$  of the standard form. It follows also from these properties, that the ratio of two densities is invariant with respect to the action of  $G$ . It will be convenient for us to represent densities in the form  $F\sqrt{\Omega}$ , where  $F$  is an invariant with respect to  $G$ , and  $\sqrt{\Omega}$  is the simplest density of rank 2 ( $\Omega$  is defined by (6)).

The parameters  $\alpha^2, \beta, \gamma$ , and  $\mu$  of the surface in the standard form are *invariants* of the surface  $\Sigma$  under the action of the group  $G$ . This means that, being expressed through the coefficients  $B_i$  and  $G_{ij}$  they are not be changed under transformations from the group  $G$ . Any other invariant can be expressed in terms of these invariants (since it can be expressed on the standard surface). We see that the total number of independent invariants (= number of independent parameters of standard surface) is equal to 4. We will use the word *basis* to denote a minimal set of invariants such that any other invariant can be represented as a function of the basic ones. It is clear that any function of invariants is an invariant itself, and two invariants are identical if they coincide on the standard surface.

It is convenient for further use to modify the basis of invariants on the standard surface:

$$\begin{aligned} p &= \alpha^2 + \mu^2, & q &= \beta^2 + \gamma^2, \\ r^2 &= \alpha^2 \mu^2, & t &= \alpha^2 \beta^2 + \gamma^2 \mu^2, \end{aligned}$$

this basis is equivalent to the original one when  $p \geq 0, q \geq 0, t \geq 0, p^2 > 4r^2$  and  $pqt \geq q^2 r^2 + t^2$ .

We will consider expressions having the form of inner products of invariant tensors  $\delta_{\alpha\beta}, \epsilon_{\alpha\beta\gamma\mu}, \epsilon^{ij}$  with covariant derivatives of  $x$  with respect to parameters  $u = u^1, v = u^2$  (or of  $X$  with respect to  $U^1$  and  $U^2$ ):

$$\begin{aligned} A_i^\alpha &= \partial_i x^\alpha|_0 = \partial_i X^\alpha|_{X_0}, & F_{ij}^\alpha &= \frac{1}{2} D_i \partial_j x^\alpha|_0 = \frac{1}{2} D_i \partial_j X^\alpha|_{X_0}, \\ i, j &= 1, 2; & \alpha &= 1, 2, 3, 4 \end{aligned}$$

(inner product is defined by means of contraction of indices, Greek indices correspond to coordinates in  $\mathbf{R}^4$ , and Roman indices denote coordinates in the 2-dimensional parameter space).  $F_{ij}^\alpha$  are covariant derivatives of  $\frac{1}{2} \partial_j x^\alpha$  which can be considered as 1-forms in parameter space for any fixed  $\alpha$ . It is easy to check that  $A_i^\alpha$  and  $F_{ij}^\alpha$  are covariant with respect to linear reparametrizations (10) and invariant with respect to quadratic reparametrizations (11); both these quantities obey vector transformation rule under the action of  $SO(4)$ . We suppose all indices in our inner products to be contracted. It is clear that such expressions transform by some one-dimensional representation of the group  $G$ . The ratio of two expressions obeying the same transformation rules is an invariant with respect to the action of  $G$ . We will form a basis of invariants having this form. To check their independence we will compare their values on the standard surface with the basic invariants obtained above. (Note, that by definition of covariant derivative the difference between  $F_{ij}$  and the usual derivative  $G_{ij}$  from formula (9) is a linear combination of the first order derivatives  $B_i = A_i$  ( $a$  and  $b$  in the form (9a)) with Christoffel symbols as coefficients. On the standard surface Christoffel symbols at  $x = 0$  are equal to zero;  $\Omega = 1$ . So, calculations on the standard surface are much simpler then in generic case.)

Since our aim at this moment is not to list all possible invariants but to find a convenient basis, we can impose some restrictions on the form of expressions sought. Let us consider the following tensors (with respect to the action of the group  $\Lambda$ ):

$$s_{ij} = \langle A_i, A_j \rangle, \quad H_{ij,kl} = \langle F_{ij}, F_{kl} \rangle, \quad T_{ij;kl,mn} = \epsilon_{\alpha\beta\gamma\delta} A_i^\alpha A_j^\beta F_{kl}^\gamma F_{mn}^\delta,$$

where  $\langle \cdot, \cdot \rangle$  denote the usual inner product in  $\mathbf{R}^4$ . Since any inner product (with respect to all indices) of some number of  $\epsilon^{ij}$  with some of these tensors transforms by some 1-dimensional representation of  $\Lambda$  and is invariant under rotations in  $\mathbf{R}^4$ , we can divide this inner product by an appropriate power of

$\Omega = \frac{1}{2}\epsilon^{ik}\epsilon^{jl}s_{ij}s_{kl}$  to get an invariant with respect to  $G = SO(4) \times \Lambda$ . The corresponding power of  $\Omega$  is defined by the following condition:

$$2\#\Omega = \#s + 2\#H + 3\#T. \quad (12)$$

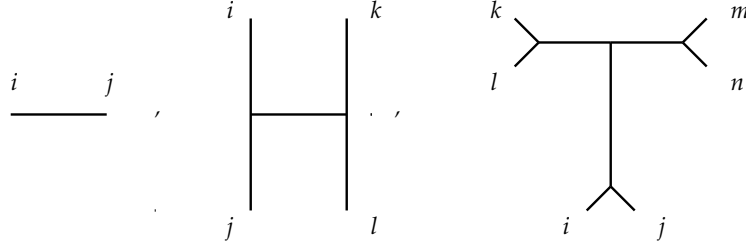
Here  $\#s$ ,  $\#H$  and  $\#T$  denote the number of appearance of the corresponding symbol in the expression,  $\#\Omega$  is the power of  $\Omega$  in the denominator. It follows from these formulas, that the number of tensors  $\epsilon^{ij}$  in the expression should be equal to  $2\#\Omega$ .

Symmetry properties of  $s, H, Y$  and  $T$  follow immediately from their definitions.

We can assign the number  $n = \#H + \#T$  to each invariant of the form above. It plays role of a quantitative measure of simplicity of the invariant and will be called the *order* of this invariant.

We would like to stress again that, being invariants, the expressions sought have the same values on every surface corresponded to the same standard form. Then, we can calculate these values on the standard surface, where it is easier.

The formulas under consideration contain many indices that makes them hardly comprehensive. To reduce this inconvenience let us introduce the following diagram technique. We assign the following (vertex) diagrams:



to  $s_{ij}$ ,  $H_{ij,kl}$ , and  $T_{ij;kl,mn}$  correspondingly.  $\epsilon^{ij}$  will be denoted by the dashed line:



And, as in many well-known diagram techniques, contraction of indices will be indicated by merging of corresponding edges at the point corresponding to the index of that contraction (we will not write contracted indices on the graph).

For example, the diagram for  $2\Omega = \epsilon^{ik}\epsilon^{jl}s_{ij}s_{kl}$  will be the following:



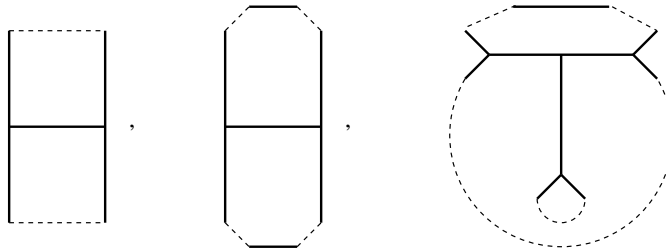
We will consider connected diagrams only. Really, if an invariant corresponds to a disconnected diagram, then it can be represented as a product of the invariants corresponding to its components, that are simpler then the original one.

Using formula (12), symmetry properties of vertices and the following equality:

$$\begin{array}{c} i \\ \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} j \\ \text{---} \end{array} = \begin{array}{c} i \quad j \\ \text{---} \end{array} \begin{array}{c} \text{---} \end{array}$$

A square diagram with solid vertical lines and dashed horizontal lines.

one can easily see that the number of independent invariants of the first order can not be greater then 3. Let us take the following three simplest diagrams of the first order:



and denote the corresponding invariants by  $Q_1$ ,  $Q_2$  and  $Q_3$ . The explicit ex-

pressions for them are:

$$Q_1 = \frac{1}{\Omega} \epsilon^{km} \epsilon^{ln} < F_{kl}, F_{mn} >,$$

$$Q_2 = \frac{1}{\Omega^2} \epsilon^{ik} \epsilon^{jn} \epsilon^{pl} \epsilon^{qm} s_{ij} s_{pq} < F_{kl}, F_{mn} >,$$

$$Q_3 = \frac{1}{\Omega^2} \epsilon^{pq} \epsilon^{ik} \epsilon^{jl} \epsilon^{mn} \epsilon_{\alpha\beta\gamma\delta} s_{ij} A_m^\alpha A_n^\beta F_{pl}^\gamma F_{kq}^\delta.$$

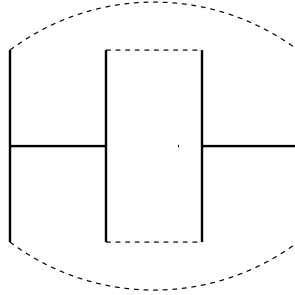
To show that they are independent one can calculate their values on the standard surface:

$$Q_1^s = 2[-\alpha^2 + \beta^2 + \gamma^2 - \mu^2] = 2[q - p],$$

$$Q_2^s = 2[\alpha^2 + \beta^2 + \gamma^2 + \mu^2] = 2[q + p],$$

$$Q_3^s = 8\alpha\mu = 8r.$$

To complete the basis we need one more invariant independent from them. Actually, any invariant with that property can be basic. To get the simplest one we will take the simplest diagram of the second order:



The fact that the corresponding invariant

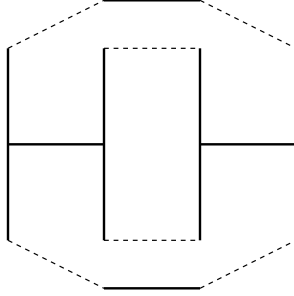
$$Q_4 = \frac{1}{\Omega^2} \epsilon^{im} \epsilon^{kp} \epsilon^{lq} \epsilon^{jn} < F_{ij}, F_{kl} > < F_{pq}, F_{mn} >$$

is independent from  $Q_1, Q_2$  and  $Q_3$  follows from its form the standard surface:

$$Q_4^s = 4[\alpha^4 + \beta^4 + \gamma^4 + \mu^4] + 8[\beta^2\gamma^2 - \alpha^2\beta^2 - \gamma^2\mu^2] = 2[q^2 + p^2 - 2r^2 - 4t].$$

So,  $Q_i$ ,  $i = 1, 2, 3, 4$  form a basis in the space of invariants, i.e. any other invariant can be represented as a function of  $Q_i$ . For example, the invariant corresponding to the diagram





can be represented as  $\frac{1}{2}Q_1Q_2 + \frac{1}{16}Q_3^2$ . (It is sufficient to check this only on the standard surface.)

The above construction proves the description of reparametrization invariant functionals on the space of surfaces in  $\mathbf{R}^4$  given in the beginning of the paper.

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### References

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